284. Probabilistic fatigue reliability assessment

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Abstract. The prediction of stochastic crack growth accumulation is important for the reliability analysis of structures as well as the scheduling of inspection and repair/replacement maintenance. Because the initial crack size, the stress, the material properties and other factors that may affect the fatigue crack growth are statistically distributed, the first-order second-moment technique is often adopted to calculate the fatigue reliability of industrial structures. In this paper, a second-order third-moment technique is presented and a three-parameter Weibull distribution is adopted to reflect the influences of skewness of the probability density function. The second-order third-moment technique that has more characteristics of those random variables that are concerned in reliability analysis is obviously more accurate than the traditional first-order second-moment technique.

Keywords: Paris–Erdogan law, fatigue reliability, Weibull distribution.

Introduction

The prediction of stochastic crack growth accumulation is important in the reliability and durability analyses of fatigue critical components. Stochastic crack growth analysis is useful for scheduling inspection and repair/replacement maintenance of structures. Various stochastic crack growth models have been proposed and studied in the literature for metallic materials and superalloys. No attempt is made herein to review the literature in this important subject area.

In practical applications of the stochastic crack growth analysis, either one of the following two distribution functions is needed: *the distribution of the crack size at any service time or the distribution of the service time to reach any given crack size*. Unfortunately, when the crack growth rate is modeled as a stochastic process, these two distribution functions are not amenable to analytical solutions, because the solution is equivalent to that of the first passage time problem. As a result, numerical simulation procedures have been used to obtain accurate results. The simulation approach is a very powerful tool, in particular with modern high-speed computers. However, in some situations, such as the preliminary analysis or design, simple approximate analytical solutions are very useful. The accuracy of such solutions depends on the sophistication of the approximation. Among attractive features of such an approximation are as follows: (i) it is mathematically simple to obtain the analytical solution for the distribution of the crack size at any service time; (ii) it is conservative in predicting the stochastic crack growth damage accumulation; (iii) it can account for the effects of variations in material crack growth resistance, usage severity and other random phenomena; and (iv) it can be implemented using a deterministic crack growth computer program.

The purpose of this paper is to present a more accurate stochastic crack growth analysis technique that is obviously more accurate than the traditional first-order second-moment technique [1] which only considers the means and variances of random variables. But probabilistic fracture mechanics problems generally involve non-normal distributions such as the lognormal, the exponential or the Weibull distribution. The skewness of a probability density function is sometimes used to measure the asymmetry of a probability density function about the mean. The second-order third-moment technique considers not only the mean and variance of a probability density function, but also the influence of skewness (which is represented by the third moment) of a probabilistic distribution. Because more characteristics of random variables are concerned with reliability analysis, the precision of reliability analysis can be increased. Particularly when random variables are not normally distributed. It is very important for industrial structures with high reliability requirement.

Paris-Erdogan Crack Growth Law as a Starting Point

The analysis of fatigue crack growth is one of the most important tasks in the design and life prediction of aircraft fatigue-sensitive structures and their components.

An example of in-service cracking from B727 aircraft (year of manufacture 1981; flight hours not available; flight cycles 39,523) [2] is given on Fig. 1.

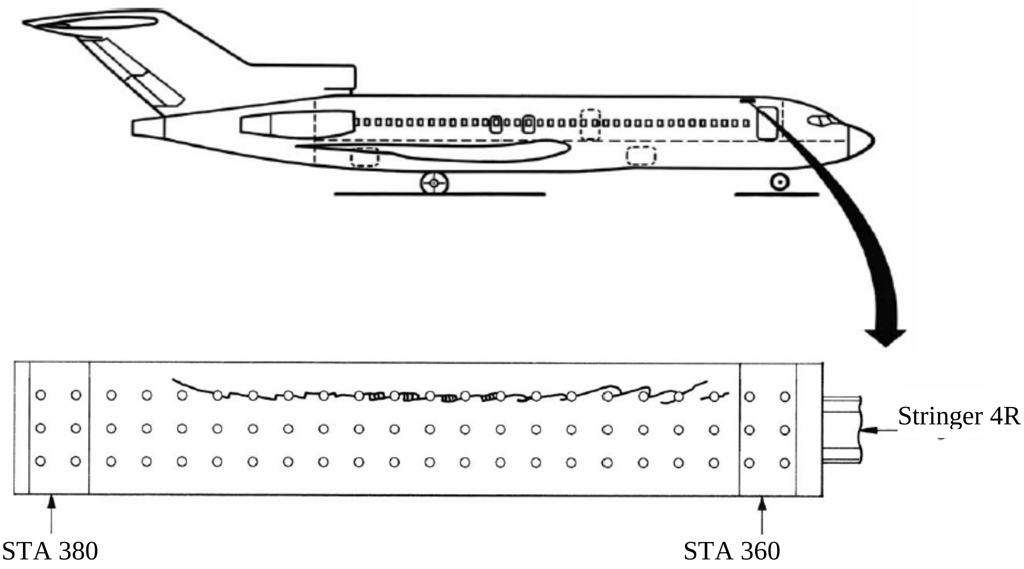

Fig. 1. Example of in-service cracking from B727 aircraft

Several models based on the principles of fracture mechanics for the prediction of fatigue crack growth in components and structures under dynamic loads have been proposed, one of the best known is the Paris–Erdogan law [3],

$$\frac{\mathrm{d}a}{\mathrm{d}N} = C(\Delta K)^{\upsilon} = C\vartheta^{\upsilon} S^{\upsilon} a^{\upsilon/2}, \qquad (1)$$

where da/dN is the crack growth rate, $\Delta K = \vartheta S\sqrt{a}$, the range of the stress intensity factor, S, the stress range, ϑ, a constant that depends on the type of load and geometry of the crack, C and υ are material constants.

If we assume all parameters in Eq. (1) are constants and deterministic, then the crack length, $a(N)$, after N cycles of stress can be obtained directly from Eq. (1). Integrating Eq. (1) from a deterministic initial crack size, $a(0)$, to crack size $a(N)$, one obtains

$$a(N) = \left([a(0)]^{1-\upsilon/2} + (1-\upsilon/2)C\vartheta^{\upsilon} S^{\upsilon} N\right)^{1/(1-\upsilon/2)}$$
$$(\upsilon \neq 2), \qquad (2)$$

$$a(N) = a(0)\exp\left(C\vartheta^2 S^2 N\right) \quad (\upsilon = 2). \qquad (3)$$

However, $a(0)$, C, ϑ and S may be all random variables with prescribed probability density functions in probabilistic fracture mechanics analysis. In practical applications of the stochastic crack growth analysis, either one of the following two distribution functions is needed: the distribution of the crack size at any given number of load cycles or the distribution of the number of load cycles to reach any given crack size.

Approximation Technique

Let us assume that the random variable Y is the function of several mutually independent random variables, X_1, X_2, …, X_n, as follows:

$$Y = f(X_1, X_2, ..., X_n), \qquad (4)$$

where vector $\boldsymbol{\mu}=(\mu_1, \mu_2, \ldots, \mu_n)$ denotes the mean of vector $\mathbf{X}=(X_1, X_2, \ldots, X_n)$, namely μ_i is the mean of random variable X_i.

To obtain a better approximation, Y is expanded about $\boldsymbol{\mu}$ to second order

$$\begin{aligned} Y = {} & f(\mu_1, \mu_2, ..., \mu_n) + \\ & + \sum_{i=1}^{n} \left(\frac{\partial Y}{\partial X_i}\right)_{\boldsymbol{\mu}} (X_i - \mu_i) + \\ & + \frac{1}{2}\sum_{i,j=1}^{n} \left(\frac{\partial^2 Y}{\partial X_i \partial X_j}\right)_{\boldsymbol{\mu}} (X_i - \mu_i)(X_j - \mu_j). \end{aligned} \qquad (5)$$

Taking the first, the second and the third moment of both sides of Eq. (5) respectively, one obtains

$$\mu_Y = f(\mu_1, \mu_2, ..., \mu_n) + \frac{1}{2}\sum_{i=1}^{n} \left(\frac{\partial^2 Y}{\partial X_i^2}\right)_{\boldsymbol{\mu}} \sigma_i^2, \qquad (6)$$

$$\sigma_Y^2 = f^2(\mu_1, \mu_2, ..., \mu_n) + \sum_{i=1}^{n}\left[\left(\frac{\partial Y}{\partial X_i}\right)^2 + Y\frac{\partial^2 Y}{\partial X_i^2}\right]_{\mu}\sigma_i^2 + \sum_{i=1}^{n}\left(\frac{\partial Y}{\partial X_i}\frac{\partial^2 Y}{\partial X_i^2}\right)_{\mu}\gamma_i - \mu_Y^2, \quad (7)$$

$$\gamma_Y = f^3(\mu_1, \mu_2, ..., \mu_n) + \frac{3}{2}\sum_{i=1}^{n}\left[2Y\left(\frac{\partial Y}{\partial X_i}\right)^2 + Y^2\frac{\partial^2 Y}{\partial X_i^2}\right]_{\mu}\sigma_i^2 + \sum_{i=1}^{n}\left[\left(\frac{\partial Y}{\partial X_i}\right)^3 + 3Y\frac{\partial Y}{\partial X_i}\frac{\partial^2 Y}{\partial X_i^2}\right]_{\mu}\gamma_i - \mu_Y^3 - 3\mu_Y\sigma_Y^2, \quad (8)$$

where μ_Y and σ_Y^2 are the mean and variance of the random variable Y, and γ_Y is the third moment of the random variable Y, which measures the amount of skewness of the distribution, σ_i^2 is the variance of the random variable X_i and γ_i is the third moment of the random variable X_i. When $\gamma > 0$, the distribution is skewed to the right such as the lognormal and exponential distributions. When $\gamma < 0$, the distribution is skewed to the left. When $\gamma = 0$, the distribution is a symmetrical distribution such as the normal distribution.

In this paper, Y may denote the crack length $a(N)$, and X_1, X_2, X_3 and X_4 may denote $a(0)$, C, ϑ and S, respectively.

Finding a Probabilistic Assessment of the Fatigue Reliability

The Weibull distribution is one of the most widely used distributions in reliability calculations. The great versatility of the Weibull distribution stems from the possibility to adjust to fit many cases where the hazard rate either increases or decreases. Further, of all statistical distributions that are available the Weibull distribution can be regarded as the most valuable because through the appropriate choice of parameters (the location parameter, the shape parameter and the scale parameter), a variety of shapes of probability density function can be modeled [4] which include the cases of $\gamma > 0$, $\gamma = 0$ and $\gamma < 0$.

The three-parameter Weibull distribution pertains to a continuous variable Y that may assume any value $\mu < y < \infty$, and is defined in terms of its density function or distribution function as follows:

$$f(y) = \frac{\delta}{\sigma}(y-\mu)^{\delta-1}\exp\left[-(y-\mu)^{\delta}/\sigma\right] \quad (y \geq \mu), \quad (9)$$

$$\Pr\{Y \leq y^{\bullet}\} = 1 - \exp\left[-(y^{\bullet}-\mu)^{\delta}/\sigma\right], \quad (10)$$

where μ is the location parameter, δ and σ are the shape parameter and the scale parameter respectively; δ, σ and μ are related to the mean value μ_Y, the variance σ_Y^2 and third moment γ_Y through the following:

$$\mu_Y = \sigma^{1/\delta}\Gamma\left(1+\frac{1}{\delta}\right) + \mu, \quad (11)$$

$$\sigma_Y^2 = \sigma^{2/\delta}\left[\Gamma\left(1+\frac{2}{\delta}\right) - \Gamma^2\left(1+\frac{1}{\delta}\right)\right], \quad (12)$$

$$\gamma_Y = \sigma^{3/\delta}\left[\begin{array}{l}\Gamma\left(1+\frac{3}{\delta}\right) - \\ -3\Gamma\left(1+\frac{2}{\delta}\right)\Gamma\left(1+\frac{1}{\delta}\right) + \\ +2\Gamma^3\left(1+\frac{1}{\delta}\right)\end{array}\right], \quad (13)$$

where $\Gamma(\cdot)$ is the gamma function.

If a variable X is normally distributed, the third moment of the random variable X is $\gamma_X = 0$ and its density function is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left[-(x-\mu)^2/(2\sigma^2)\right]. \quad (14)$$

Then the mean and variance of the random variable X are given by

$$\mu_X = \mu, \quad (15)$$

$$\sigma_X^2 = \sigma^2. \quad (16)$$

If a variable X is exponentially distributed and its density function is given by

$$f(x) = \sigma^{-1}\exp\left[-(x-\mu)/\sigma\right], \quad (17)$$

then the mean, the variance and the third moment of the variable is given by

$$\mu_X = \mu + \sigma, \tag{18}$$

$$\sigma_X^2 = \sigma^2, \tag{19}$$

$$\gamma_X = 2\sigma^3. \tag{20}$$

If a variable X is lognormally distributed and its density function is given by

$$f(x) = \frac{1}{\sigma x\sqrt{2\pi}} \exp\left[-(\ln x - \mu)^2/(2\sigma^2)\right], \tag{21}$$

then the mean, the variance and the third moment of the variable is given by

$$\mu_X = \exp(\mu + 0.5\sigma^2), \tag{22}$$

$$\sigma_X^2 = \exp(2\mu + \sigma^2)\left[\exp(\sigma^2) - 1\right], \tag{23}$$

$$\begin{aligned}\gamma_X &= \exp(3\mu + 4.5\sigma^2) - \\ &- 3\exp(3\mu + 2.5\sigma^2) + \\ &+ 2\exp(3\mu + 1.5\sigma^2).\end{aligned} \tag{24}$$

If a variable X follows the Weibull distribution and its density function is defined by Eq. (9), then the mean, the variance and the third moment of the variable X is calculated from Eqs. (11)–(13).

Thus, procedure for finding a probabilistic assessment of the fatigue reliability can be summarized as follows. If, $a(0)$, C, ϑ and S in Eqs. (2) and (3) may be all random variables with prescribed probability density functions such as the normal, the lognormal, the exponential and the Weibull distributions, in which the means, the variances and the third moments can be calculated from Eqs. (14)–(24) (the Weibull distribution calculated from Eqs. (11)–(13)), then the mean, the variance and the third moment of the crack length $a(N)$ at any given N cycles of stress can be calculated from Eqs. (6)–(8). The parameters δ, σ and μ of the Weibull distribution are solved from Eqs. (11)–(13) and the reliability of the cracked structure after N cycles of stress is obtained from Eq. (10).

Numerical Example

Let us assume that a cracked structure with the fatigue crack growth rates $da/dN = 5 \times 10^{-5}$ mm/cycle is researched. The probability density function of the initial crack size $a(0)$ is exponential distribution function,

$$f(x)=\exp[-(x-2)]. \tag{25}$$

The critical crack length is $a^{\bullet}$=10 mm. The reliability of the cracked structure after N=10^5 cycles of stress is obtained as follows:

(i) Suppose that the constant of the Paris–Erdogan law is υ=0 and only $a(0)$ is random variable. From Eq. (2), one can obtain the crack size after N=10^5 cycles of stress:

$$a(N) = a(0) + 5\times10^{-5} N = a(0) + 5 \text{ (mm)}. \tag{26}$$

(ii) From Eqs. (17)–(20), the mean, the variance and the third moment of the initial crack size $a(0)$ is calculated in which σ=1, μ=2 mm. Then from Eqs. (6)–(8), one can obtain the mean, the variance and the third moment of the crack size a after N=10^5 cycles of stress as follows:

$$\mu_{a(N)} = 8.0, \quad \sigma_{a(N)}^2 = 1.0, \quad \gamma_{a(N)} = 2.0. \tag{27}$$

(iii) Substituting $\mu_{a(N)}$, $\sigma_{a(N)}^2$, and $\gamma_{a(N)}$ into Eq. (11), Eq. (12) and Eq. (13), respectively, these simultaneous equations for δ, σ and μ can be solved to obtain three parameters of the Weibull distribution as follows:

$$\delta=1.0, \quad \sigma=1.0, \quad \mu=7.0. \tag{28}$$

(iv) As the density distribution of the crack length $a(N)$ is defined as Eq. (9), the reliability of the cracked structure after N=10^5 cycles of stress is calculated by Eq. (10) as follows:

$$\begin{aligned}&\Pr\{a(N) \le a^{\bullet}\} = \\ &= 1 - \exp\left[-(a^{\bullet} - \mu)^{\delta}/\sigma\right] = \\ &= 1 - \exp[-(10-7)] = 0.95.\end{aligned} \tag{29}$$

Let us assume that we use the above technique to calculate the reliability of the cracked structure, where only the mean $\mu_{a(N)} = 8.0$ and the variance $\sigma_{a(N)}^2 = 1.0$ of the crack size $a(N)$ are considered. The distribution of the crack size $a(N)$ is assumed to be a normal distribution. One can obtain

$$\begin{aligned}&\Pr\{a(N) \le a^{\bullet}\} = \Phi\left(\frac{10-8}{1}\right) = \Phi(2) = \\ &= 0.9772.\end{aligned} \tag{30}$$

where $\Phi(\cdot)$ is the standard normal distribution function.

Actually the problem discussed has an analytical solution, which is given as follows:

$$\begin{aligned}&\Pr\{a(N) \le a^{\bullet}\} = \Pr\{a(0) + 5 \le 10\} = \\ &= \Pr\{a(0) \le 5\} =\end{aligned}$$

$$= \int_{2.}^{5.} \exp\left[-(x-2)\right] dx =$$
$$= 1 - \exp\left[-(5-2)\right] = 0.95, \tag{31}$$

where the result is the same as the one of (29).

Conclusion

From the case discussed above, the second-order third-moment technique is obviously more accurate than the traditional first-order second-moment technique.

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